MULTIFRACTAL ANALYSIS OF NON-UNIFORMLY HYPERBOLIC SYSTEMS

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Abstract. We prove a multifractal formalism for Birkhoff averages of continuous functions in the case of some non-uniformly hyperbolic maps, which includes interval examples such as the Manneville–Pomeau map.

1. Introduction and Notation

In this paper we look at the multifractal analysis of some non-uniformly hyperbolic maps. In particular we look at the problem for the Birkhoff averages of continuous functions. This type of problem is well understood in the hyperbolic case (see [2],[10],[12] for specific results and [11] for an introduction to the subject). However in the non-uniformly hyperbolic case much less is known. The results known so far concerning Hausdorff dimension of such spectra are limited to Lyapunov spectra ([4],[9],[8]) and local dimension of Gibbs' measures ([3]). Furthermore the methods cannot be applied to Birkhoff averages for general continuous functions. In the case of general continuous functions there are results for topological entropy [13], but not for Hausdorff dimension. See also [1] for some work on parabolic horseshoes. Finally there is work for lcoal dimensions for countable state systems, [5] which can be related to parabolic systems through inducing schemes. In this paper we produce results for the Hausdorff dimension which extend some of the results of [10] into the non-uniformly hyperbolic setting.

We begin with a classical example. Let $T : [0,1] \to [0,1]$ be the Manneville–Pomeau map defined by $Tx = x + x^{1+\beta} \mod 1$, where $0 < \beta < 1$. Let $f : [0,1] \to \mathbb{R}$ be continuous and define

$$\Lambda_\alpha = \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \alpha \right\}.$$

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Let us denote, $\alpha_{\min} = \inf_{\mu \in \mathcal{E}}\{\int f \, \mathrm{d}\mu\}$ and $\alpha_{\max} = \sup_{\mu \in \mathcal{E}}\{\int f \, \mathrm{d}\mu\}$, where $\mathcal{E} = \mathcal{E}_T([0,1])$ denotes the space of T-invariant ergodic probability measures.

For $\alpha \in [\alpha_{\min}, \alpha_{\max}] \backslash \{f(0)\}$ we have

$$\dim_H \Lambda_\alpha = \sup_{\mu \in \mathcal{M}_T([0,1])} \left\{ \frac{h(\mu, T)}{\int \log T'(x) \, \mathrm{d}\mu} : \int f \, \mathrm{d}\mu = \alpha \right\},$$

where $\mathcal{M}_T([0,1])$ denotes the T-invariant probability measures. We can also show that $\dim_H \Lambda_{f(0)} = 1$.

We can consider the related problem for iterated function systems. Let $T_i : [0,1] \to [0,1]$, $1 \leq i \leq m$, be C^1 maps such that $T_i'(x) > 0$ and for distinct i, j, we have $T_i(0,1) \cap T_j(0,1) = \emptyset$. At this stage the only additional assumption we make is that $\operatorname{diam}(T_{i_1} \circ \cdots \circ T_{i_n}([0,1]))$ converges to 0, uniformly in all sequences of maps.

Let $\mathcal{A} = \{1, \ldots, m\}$ and let $\Sigma = \mathcal{A}^{\mathbb{N}}$ be the one-sided shift space on m symbols, $\sigma : \Sigma \to \Sigma$ the usual shift map and let $f : \Sigma \to \mathbb{R}$ be a continuous function. Given $n \geq 1$ we let $A_n f(\omega) = \frac{1}{n}\sum_{i=0}^{n-1} f(\sigma^i \omega)$ denotes the nth level Birkhoff average of the function $f : \Sigma \to \mathbb{R}$. Let $\Pi : \Sigma \to [0,1]$ be the natural projection defined by

$$\Pi(\omega) = \lim_{n \to \infty} T_{\omega_1} \circ \cdots \circ T_{\omega_n}(0), \quad \omega \in \Sigma.$$

Furthermore we can define the attractor of the system by $\Lambda = \Pi(\Sigma)$. Note that by a fixed point in the iterated function system we mean the projection of a fixed point in the one-sided shift space. We will consider the sets

$$X_\alpha = \left\{ \omega \in \Sigma : \lim_{n \to \infty} A_n f(\omega) = \alpha \right\}$$

and their images $\Pi(X_\alpha) \subseteq \Lambda$.

Let $\tilde{\Sigma}$ be the subset of Σ on which the diameters tend to 0 exponentially, i.e., let $\tilde{\Sigma} = \{\omega : \liminf_{n \to \infty} A_n g(\omega) > 0\}$, where $g(\omega) := -\log \left| T_{\omega_1}'(\Pi(\omega)) \right|$. We introduce the notation $\alpha_{\min} = \inf_{\mu \in \mathcal{E}_\sigma(\Sigma)}\{\int f \, \mathrm{d}\mu\}$ and $\alpha_{\max} = \sup_{\mu \in \mathcal{E}_\sigma(\Sigma)}\{\int f \, \mathrm{d}\mu\}$. Let $\mathcal{M}_\sigma(\Sigma)$ denote the σ-invariant measures and let $\mathcal{E}_\sigma(\Sigma)$ denote the σ-ergodic measures. Denote the entropy of $\mu \in \mathcal{M}_\sigma(\Sigma)$ by $h(\mu, \sigma)$ and the Lyapunov exponent by $\lambda(\mu, \sigma^n)$. We can now state our first result as follows.

Theorem 1. *For $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ we have*

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} : \int f \, \mathrm{d}\mu = \alpha \text{ and } \lambda(\mu, \sigma) > 0 \right\}.$$

We now consider the specific prototype case where $T_1, \ldots, T_m : [0,1] \to [0,1]$ are C^1 maps with fixed points $(x_1, \ldots, x_m)$ such that $T_i(x_i) = x_i$, where $x_i = \Pi(i, i, i, \ldots)$, $i \in \mathcal{A}$. We assume that $T_i'(x_i) \leq 1$ and $0 < T_i'(x) < 1$ everywhere else, and for

distinct i, j, we have $T_i(0,1) \cap T_j(0,1) = \emptyset$. Furthermore we assume that for any $\epsilon > 0$ there exists a $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $\lambda(\mu, \sigma) > 0$ and $\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} \geq \dim_H \Lambda - \epsilon$. In other words we have a system with a finite number of parabolic fixed points with hyperbolic measures with dimension arbitrarily close to that of the attractor. If the maps T_i are all C^{1+s} for some $s > 0$ then we can deduce from Theorem 4.6 in [14] that this condition is satisfied. This is reminiscent of Katok's result on approximation in topological entropy by hyperbolic horseshoes, [7]. We can use Theorem 1 to deal with the cases where the Lypaunov exponent is nonzero. We use a method similar to the work of Gelfert and Rams [4] to deal with the cases where the Lyapunov exponent can be zero. Let $\mathcal{I} \subset \mathcal{A}$ represent the set of *indifferent fixed points* so that $T_i'(x_i) = 1$ whenever $i \in \mathcal{I}$. For $i \in \mathcal{I}$, let $\alpha_i = f(i, i, i, \ldots)$ and let

$$A = [\min_{i \in \mathcal{I}} \{\alpha_i\}, \max_{i \in \mathcal{I}} \{\alpha_i\}]. \tag{1.1}$$

Theorem 2. *Assume that the iterated function system has a finite number of indifferent fixed points as above and that for any $\epsilon > 0$ there exists a measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ with $\lambda(\mu, \sigma) > 0$ and $\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} \geq \dim_H \Lambda - \epsilon$. Then, for $\alpha \in [\alpha_{min}, \alpha_{max}] \setminus A$ we have*

$$\dim_H \Pi(X_\alpha) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} : \int f \, \mathrm{d}\mu = \alpha \right\},$$

and $\dim_H \Pi(X_\alpha) = \dim_H \Lambda$ for all $\alpha \in A$.

It is straightforward to deduce that in this case $\dim_H \Pi(X_\alpha)$ is a continuous function of α with the possible exception of the endpoints of A:

Corollary 1. *The function $r : [\alpha_{\min}, \alpha_{\max}] \to \mathbb{R}$ defined by $r(\alpha) = \dim_H \Pi(X_\alpha)$ is constant in the interior of A and continuous in $[\alpha_{\min}, \alpha_{\max}] \backslash A$.*

Proof. Since r is clearly constant in the interior of A we just consider $\alpha \in [\alpha_{\min}, \alpha_{\max}] \backslash A$. To start let $\mu_1, \mu_2 \in \mathcal{M}_\sigma(\Sigma)$ such that $\int f \, \mathrm{d}\mu_1 = \alpha_{\min}$ and $\int f \, \mathrm{d}\mu_2 = \alpha_{\max}$. For $\alpha \in [\alpha_{\min}, \alpha_{\max}] \backslash A$ consider a sequence $\{\beta_n\}_{n \in \mathbb{N}}$ such that $\beta_n \to \alpha$ as $n \to \infty$. It follows that $r(\alpha) \geq \limsup_{n \to \infty} r(\beta_n)$ by upper semicontinuity of entropy (see Theorem 8.2 [15]). For $\epsilon > 0$ we let μ satisfy $\int f \, \mathrm{d}\mu = \alpha$ and $\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} > r(\alpha) - \epsilon$. By considering measures ν_n such that $\int f \, \mathrm{d}\nu_n = \beta_n$ of the form $\nu_n = p_n \mu_1 + (1 - p_n)\mu$ or $\nu_n = p_n \mu_2 + (1 - p_n)\mu$ for appropriate $p_n \searrow 0$, we can deduce that $r(\alpha) \leq \liminf_{n \to \infty} r(\beta_n)$. □

These results are well understood in the case of uniformly contracting systems (see [12],[10],[2]). The novelty in this work is that we can analyse certain non-uniformly hyperbolic systems. Moreover, our methods do not involve either thermodynamic

formalism or the use of large deviation theory. For an introduction to dimension theory and multifractal analysis the reader is referred to [11]. All the necessary definitions and results from ergodic theory can be found in [15].

We can also use Theorem 2 to deduce a result which applies to non-uniformly expanding maps of the interval (such as the Manneville–Pomeau map mentioned earlier).

Corollary 2. *Let $T : [0,1] \to [0,1]$ be a piecewise onto C^1 map with a finite number of parabolic fixed points x_i such that $T(x_i) = x_i$ and $T'(x_i) = 1$ but $T'(x) > 1$ for $x \in [0,1] \backslash \cup_i x_i$. We also assume the existence of a hyperbolic measure with dimension arbitrarily close to 1. Let $f : [0,1] \to \mathbb{R}$ be continuous and let*

$$\Lambda_\alpha = \{x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \alpha\}.$$

If we let $A = [\min_i\{f(x_i)\}, \max_i\{f(x_i)\}]$ then for $\alpha \in [\alpha_{\min}, \alpha_{\max}] \backslash A$ we have

$$\dim_H \Lambda_\alpha = \sup_{\mu \in \mathcal{M}_T([0,1])} \left\{ \frac{h(\mu,T)}{\int \log T'(x) \, \mathrm{d}\mu(x)} : \int f \, \mathrm{d}\mu = \alpha \right\}.$$

We also have that for all $\alpha \in A$

$$\dim_H \Lambda_\alpha = 1.$$

Proof. This follows by noting that Theorem 2 can be applied to the iterated function system defined by the inverse branches of this map. □

Without the assumption of the existence of a hyperbolic measure of dimension arbitrarily close to 1 the result would be the same except that we would no longer have equality for $\alpha \in A$ but that the dimension is bigger than the supremum of the dimension of hyperbolic measures and less than the dimension of the attractor. We don't know of any examples where this situation occurs. It is also possible to generalise the result to Markov maps however we just work in the Bernoulli case to ease the exposition. It follows from Theorem 4.6 in [14] that for any such system where the inverse branches are C^{1+s} for $s > 0$ that this hypothesis is satisfied. We now give some examples to illustrate this corollary and the difference of the result from the expanding case.

Example 1. The Manneville–Pomeau map is known to have a finite T-invariant absolutely continuous probability measure (we denote this measure by μ_{SRB}) in the case when $0 < \beta < 1$. For $\beta \geq 1$ there is no T-invariant absolutely continuous probability measure but there are measures of dimension arbitrarily close to 1. So provided $\beta > 0$ it satisfies the hypotheses of the Corollary 2. Let $f : [0,1] \to \mathbb{R}$ be

a continuous function such that $\int f \, \mathrm{d}\mu_{SRB} > f(0) = \alpha_{\min}$. Then for $\beta \in (0,1)$ we have that

$$\dim_H \Lambda_\alpha \begin{cases} = 1 & \text{for} \quad \alpha \in \left[f(0), \int f \, \mathrm{d}\mu_{SRB}\right] \\ < 1 & \text{for} \quad \alpha \in \left[\int f \, \mathrm{d}\mu_{SRB}, \alpha_{\max}\right] \end{cases}$$

In the case where $\beta > 1$ we have that $\dim_H \Lambda_{f(0)} = 1$ and $\dim_H \Lambda_\alpha < 1$ for $\alpha \in (f(0), \alpha_{\max})$.

Example 2. Let $T : [0,1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} \frac{x}{1-x} & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ \frac{2x-1}{x} & \text{for} \quad \frac{1}{2} < x \leq 1 \end{cases}.$$

Thus T has parabolic fixed points at 0 and 1 but is expanding everywhere else. There are no absolutely continuous T-invariant probability measures but there are T-invariant measures of dimension arbitrarily close to 1. Hence for any $f : [0,1] \to \mathbb{R}$ which is continuous we can apply Corollary 2. In the case where f is monotone increasing then $A = [f(0), f(1)] = [\alpha_{\min}, \alpha_{\max}]$ and $\dim_H \Lambda_\alpha = 1$ for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

The layout of the rest of the paper is as follows. In the next three sections we give the proof of Theorem 1. In the final section we go on to deduce Theorem 2.

2. Preliminary Results

In this section we prove the basic lemmas needed to prove Theorem 1. These involve an approximation result and methods of invariant and ergodic invariant measures. We will use $\mathcal{M}_{\sigma^n}(\Sigma)$, and $\mathcal{E}_{\sigma^n}(\Sigma)$ to denote the σ^n-invariant and σ^n-ergodic measures respectively. For $\mu \in \mathcal{M}_{\sigma^n}(\Sigma)$ let $h(\mu, \sigma^n)$ denote the entropy of μ with respect to σ^n. Let $\mathcal{F}_n$ be the finite algebra generated by the n-cylinders. Since, for each n, $\mathcal{F}_n$ is a generating partition for σ^n, we have for $\mu \in \mathcal{M}_{\sigma^n}(\Sigma)$ that $h(\mu, \sigma^n) = nh(\mu, \sigma)$ where

$$h(\mu, \sigma) = \lim_{N \to \infty} \frac{1}{N} H\left(\mu|_{\mathcal{F}_N}\right)$$

and $H(\nu)$ denotes the Shannon entropy

$$H(\nu) = \sum_{A \in \mathcal{A}} \nu(A) \log(1/\nu(A)),$$

defined for probabilities ν on finite algebras $\mathcal{A}$.

Let $g : \Sigma \to \mathbb{R}$ be defined by $g(\omega) = -\log T'_{\omega_1}(\Pi(\sigma(\omega)))$. We then denote the Lyapunov exponent of a measure $\mu \in \mathcal{M}_{\sigma^n}(\Sigma)$ by

$$\lambda(\mu, \sigma^n) = \int \sum_{k=0}^{n-1} g(\sigma^k \omega)\, \mathrm{d}\mu(\omega).$$

Let $\pi_n : \Sigma \to \mathcal{A}^n$ denote the natural projection onto the first n symbols, i.e. $\pi_n(\omega) \mapsto (\omega_1, \ldots, \omega_n) \in \mathcal{A}^n$. For cylinder sets we use the notation $[\alpha_1, \ldots, \alpha_n]$ for $\pi_n^{-1}(\alpha_1, \ldots, \alpha_n)$ and for $\omega \in \Sigma$ we let $[\omega]_n$ denote $\pi_n^{-1}(\pi_n(\omega))$. Let $\mathcal{F}_n$ be the finite σ-algebra generated by the n-cylinders $\{[\omega]_n : \omega \in \Sigma\}$.

For a function $f : \Sigma \to \mathbb{R}$, define the nth variation as

$$\operatorname{var}_n f = \sup_{[\omega']_n = [\omega]_n} |f(\omega) - f(\omega')|.$$

By definition, $\lim_{n\to\infty} \operatorname{var}_n f = 0$ if f is continuous and then also $\lim_{n\to\infty} \operatorname{var}_n A_n f = 0$.

Let $I_n(\omega) \subset I$ denote the interval $T_{\omega_1} \circ \cdots \circ T_{\omega_n}([0,1])$, where $I_0(\omega) \equiv [0,1]$, and let the corresponding diameters be given by $D_n(\omega) = \operatorname{diam}(I_n(\omega))$. For $n \geq 1$, we write $\tilde{\lambda}_n(\omega)$ for $-\frac{1}{n} \log D_n(\omega)$.

We start by showing that, for lage n, $\tilde{\lambda}_n$ is well approximated by the Birkhoff average $A_n g(\omega)$.

Lemma 1. *Let $T_i : [0,1] \to [0,1]$, $1 \leq i \leq m$, be C^1 maps such that $T'_i(x) > 0$ and such that for distinct i, j, we have $T_i(0,1) \cap T_j(0,1) = \emptyset$. In addition we assume that $D_n(\omega) \to 0$, uniformly in ω. Then*

$$\lim_{n\to\infty} \sup_{\omega \in \Sigma} \{| -\frac{1}{n} \log D_n(\omega) - A_n g(\omega)|\} = 0$$

Proof. We introduce the functions $g_n : \Sigma \to \mathbb{R}$, $n \geq 1$, defined by

$$g_n(\omega) = -\log \frac{D_n(\omega)}{D_{n-1}(\sigma\omega)}.$$

It is immediate from the definitions that

$$-\log D_n(\omega) = \sum_{i=0}^{n-1} g_{n-i}(\sigma^i \omega).$$

We can relate this identity to the Birkhoff averages of $g : \Sigma \to \mathbb{R}$ using the following fact

$$g_n(\omega) \to g(\omega) \quad \text{uniformly in } \omega \text{ as } n \to \infty. \tag{2.1}$$

To see (2.1), we note that for $n \geq 2$,

$$g_n(\omega) = -\log\left(\frac{1}{D_{n-1}(\sigma\omega)}\int_{I_{n-1}(\sigma\omega)} T'_{\omega_1}(x)\,dx\right) = -\log T'_{\omega_1}(\xi)$$

for some $\xi \in I_{n-1}(\sigma\omega)$ by the intermediate value theorem. By hypothesis, each $\log T'_i(x)$ is (uniformly) continuous and thus, since $\operatorname{diam}(I_{n-1}(\sigma\omega)) = D_{n-1}(\sigma\omega)$ tends to 0 uniformly, it follows that

$$g_n(\omega) - g(\omega) = \log T'_{\omega_1}(\Pi(\sigma\omega)) - \log T'_{\omega_1}(\xi)$$

also tends to 0 uniformly as $n \to \infty$.

Let $\epsilon > 0$ and note that by (2.1), we can choose N_1 such that for $n \geq N_1$ and $\omega \in \Sigma$ we have $|g_n(\omega) - g(\omega)| \leq \frac{\epsilon}{2}$. We can also find a $C_1 > 0$ where $|g_n(\omega) - g(\omega)| < C_1$ for all $\omega \in \Sigma$. Let $N = \left(\left[\frac{2C_1}{\epsilon}\right] + 1\right) N_1$.

For $n \geq N$ and $\omega \in \Sigma$ we have that

$$\begin{aligned}
-\log D_n(\omega) &= \sum_{i=0}^{n-1} g_{n-i}(\sigma^i\omega) \\
&\leq \sum_{i=0}^{n-1-N_1} \left(g(\sigma^i\omega) + \frac{\epsilon}{2}\right) + \sum_{i=n-N_1}^{n-1} g_{n-i}(\sigma^i\omega) \\
&\leq \sum_{i=0}^{n-1} g(\sigma^i\omega) + (n-N_1)\frac{\epsilon}{2} + C_1N_1 \\
&\leq \sum_{i=0}^{n-1} g(\sigma^i\omega) + (n-N)\frac{\epsilon}{2} + N\frac{\epsilon}{2} \text{ (since } C_1N_1 \leq \frac{N\epsilon}{2}) \\
&\leq \sum_{i=0}^{n-1} \left(g(\sigma^i(\omega)) + \epsilon\right).
\end{aligned}$$

The other inequality is similar. $\square$

We now need to relate σ^n-ergodic measures to σ-ergodic measures and σ-invariant measures to σ^n-ergodic measures. Given $\nu \in \mathcal{E}_{\sigma^n}(\Sigma)$ we define $\mu = A_n^*\nu$ as the measure

$$\mu = \frac{1}{n}\sum_{k=0}^{n-1} \nu \circ \sigma^{-k}.$$

Lemma 2. *If $\mu = A_n^*\nu$, $\nu \in \mathcal{E}_{\sigma^n}(\Sigma)$, then $\mu \in \mathcal{E}_\sigma(\Sigma)$ and*

(1) $h(\mu, \sigma) = \frac{1}{n}h(\nu, \sigma^n)$.
(2) $\lambda(\mu, \sigma) = \frac{1}{n}\lambda(\nu, \sigma^n)$.

(3) $\int f \, \mathrm{d}\mu = \int A_n f \, \mathrm{d}\nu$.

Proof. The first part of this lemma follows by Abramov's Theorem (see [15] Theorem 4.13). The final two parts are routine calculations. □

The next lemma states that we may approximate any invariant measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ by ergodic measures in $\mathcal{E}_{\sigma^n}(\Sigma)$. A probability measure μ on Σ is *nth level Bernoulli* if the *n-blocks* $\pi_n \circ T^{kn}(\omega)$ are independent and identically distributed for $k = 0, 1, \ldots$. An nth level Bernoulli measure is always σ^n-invariant and ergodic with respect to σ^n. Moreover, we have a natural continuous bijection $\nu \mapsto \nu^{\otimes}$ between probabilities on blocks $\nu \in \mathcal{M}(\mathcal{A}^n)$ and the corresponding nth level Bernoulli measures. The block probability ν is the marginal of the corresponding nth level Bernoulli measure, i.e. $(\nu^{\otimes}) \circ \pi_n^{-1} = \nu$, and we have $h(\nu^{\otimes}, \sigma^n) = H(\nu)$.

Lemma 3. *For any $\mu \in \mathcal{M}_\sigma(\Sigma)$, we can find a sequence of measures $\{\mu_n\}$ converging to μ in the weak*-topology such that*

(1) *μ_n is nth level Bernoulli,*
(2) $\lim_{n\to\infty} \frac{1}{n} h(\mu_n, \sigma^n) = h(\mu, \sigma)$ *and* $\lim_{n\to\infty} \frac{1}{n} \lambda(\mu_n, \sigma^n) = \lambda(\mu, \sigma)$,

and moreover, if $\int f \, \mathrm{d}\mu = \alpha \in (\alpha_{\min}, \alpha_{\max})$, then we may in addition assume that

(3) $\int A_n f \, \mathrm{d}\mu_n = \alpha$.

Proof of Lemma 3. To see the first part, let $\mu_n = (\mu \circ \pi_n^{-1})^{\otimes}$. Then $\mu_n|_{\mathcal{F}_n} = \mu|_{\mathcal{F}_n}$ and, since $\mathcal{F}_n$ increases to the Borel σ-algebra, this implies that $\mu_n \to \mu$ in the weak*-topology. Also,

$$\lambda(\mu, \sigma) = \lim_{n\to\infty} \int A_n g \, \mathrm{d}\mu_n \text{ and } \alpha = \int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int A_n f \, \mathrm{d}\mu_n$$

and by definition we have

$$h(\mu, \sigma) = \lim_{n\to\infty} \frac{1}{n} H(\mu \circ \pi_n^{-1}) = \lim_{n\to+\infty} \frac{1}{n} h(\mu_n, \sigma^n).$$

We need to work a little bit more to modify this construction to give a sequence $\tilde{\mu}_n$ of nth level Bernoulli measures that also satisfies (3), in addition to (1) and (2). Without loss of generality, we may assume that $\int A_{n_j} f \, \mathrm{d}\mu_{n_j} \leq \alpha$ for some infinite sequence $\mathcal{N} = \{n_j\}$. We proceed to construct $\tilde{\mu}_n$ for such n's and a symmetric construction gives $\tilde{\mu}_n$ for $n \in \mathbb{N} \setminus \mathcal{N}$.

By the ergodic theorem, we can always find a point $x \in \Sigma$, a number $0 < \rho < (\alpha_{max} - \alpha)/3$ and an integer $N > 0$ such that $A_n f(x) \geq \alpha + 3\rho$ for all $n \geq N$. Denote by $\nu_n = \mu_n \circ \pi_n^{-1} \in \mathcal{M}(\mathcal{A}^n)$ the block marginal of μ_n and let $\delta_n \in \mathcal{M}(\mathcal{A}^n)$ denote

the Dirac measure at the word $(x_1, x_2, \ldots, x_n)$. Then $\delta_n^{\otimes}$ is a Dirac measure on the corresponding periodic sequence, and we can also assume that $\int A_n f \, \mathrm{d}(\delta_n^{\otimes}) > \alpha + 2\rho$ for all $n \geq N$.

Furthermore, define for $s \in [0,1]$ a nth level Bernoulli measure

$$\xi_{s,n} := (s\nu_n + (1-s)\delta_n)^{\otimes}.$$

Note that the map $s \mapsto F(s) = \int A_n f \, \mathrm{d}\xi_{s,n}$ is continuous and hence, since $n \in \mathcal{N}$ means that $F(1) \leq \alpha$ and $F(0) > \alpha + 2\rho$, we deduce that $\int A_n f \, \mathrm{d}\xi_{s_n,n} = \alpha$, for some $s_n \in [0,1]$.

We need to show that we can choose $s_n \to 1$ as n tends to infinity with $n \in \mathcal{N}$, since then by setting $\tilde{\mu}_n = \xi_{s,n}$ we have

$$\frac{1}{n} h(\tilde{\mu}_n, \sigma^n) = s_n \frac{1}{n} h(\mu_n, \sigma^n) + (1 - s_n) \cdot 0 + O(1/n) \to h(\mu, \sigma)$$

and

$$\lambda(\tilde{\mu}_n, \sigma) = s_n \cdot \lambda(\mu_n, \sigma) + (1 - s_n) \cdot \int A_n g \, \mathrm{d}(\delta_n^{\otimes}) + O(\mathrm{var}_n\, A_n g) \to \lambda(\mu, \sigma)$$

as required. The fact that $\lim_{n\to\infty} \mathrm{var}_n\, A_n g = 0$ follows from the uniform continuity of g.

For all $s \in [0,1]$, the n-block marginals for $\xi_{s,n}$ and

$$\zeta_{s,n} := s\nu_n^{\otimes} + (1-s)\delta_n^{\otimes}$$

coincide, i.e. $\xi_{s,n} \circ \pi_n^{-1} = \zeta_{s,n} \circ \pi_n^{-1}$. Therefore

$$\int A_n f \, \mathrm{d}\xi_{s,n} \geq \int A_n f \, \mathrm{d}\zeta_{s,n} - \mathrm{var}_n\, A_n f.$$

and since $\nu_n^{\otimes} = \mu_n$ we deduce

$$\begin{aligned} \int A_n f \, \mathrm{d}\zeta_{s,n} &= s \int A_n f \, \mathrm{d}\mu_n + (1-s) \int A_n f \, \mathrm{d}(\delta_n^{\otimes}) \\ &\geq \alpha - s\epsilon_n + (1-s) \cdot (2\rho), \end{aligned}$$

where $\epsilon_n = \alpha - \int A_n f \, \mathrm{d}\mu_n$.

Thus

$$\begin{aligned} \alpha &= \int A_n f \, \mathrm{d}\tilde{\mu}_n \\ &\geq \alpha - s\epsilon_n + (1-s) \cdot (2\rho) - \mathrm{var}_n\, A_n f, \end{aligned}$$

which, since $\epsilon_n \to 0$ and $\mathrm{var}_n\, A_n f \to 0$ as $n \in \mathcal{N}$ tends to infinity, gives that $\lim_{n\to+\infty} s_n = 1$. $\square$

3. Lower Bound

In this section we shall prove the lower bound in Theorem 1, i.e.

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) \geq \sup_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} : \int f \, \mathrm{d}\mu = \alpha, \lambda(\mu,\sigma) > 0 \right\}. \tag{3.1}$$

We start by calculating the dimension of the projection of any invariant measure $\mu \in \mathcal{E}_{\sigma^n}(\Sigma)$ with positive Lypaunov exponent.

Lemma 4 (Hofbauer–Raith)**.** *Let $\mu \in \mathcal{M}_{\sigma^n}(\Sigma)$ be ergodic with respect to σ^n and satisfy $\lambda(\mu, \sigma^n) > 0$. We have that*

$$\dim_H(\mu \circ \Pi^{-1}) = \frac{h(\mu,\sigma^n)}{\lambda(\mu,\sigma^n)}.$$

Proof. This was originally shown by Hofbauer and Raith in [6]. The proof can be seen by applying Lemma 1, together with the Birkhoff Ergodic Theorem and the Shannon–McMillan–Breiman Theorem. □

Let $\mu \in \mathcal{E}_{\sigma^n}(\Sigma)$ satisfy both $\int A_n f \, \mathrm{d}\mu = \alpha$ and $\lambda(\mu, \sigma^n) > 0$. It follows from the Birkhoff Ergodic Theorem that for any such μ we have $\mu(X_\alpha \cap \tilde{\Sigma}) = 1$. Hence we can deduce from Lemma 4 that

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) \geq \frac{h(\mu,\sigma^n)}{\lambda(\mu,\sigma^n)}$$

and so

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) \geq \sup_{\mu \in \mathcal{E}_{\sigma^n}(\Sigma)} \left\{ \frac{h(\mu,\sigma^n)}{\lambda(\mu,\sigma^n)} : \int A_n f \, \mathrm{d}\mu = \alpha \text{ and } \lambda(\mu,\sigma^n) > 0 \right\}.$$

We can now apply Lemma 3 to see that

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) \geq \sup_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} : \int f \, \mathrm{d}\mu = \alpha \text{ and } \lambda(\mu,\sigma) > 0 \right\}$$

for $\alpha \in (\alpha_{\min}, \alpha_{\max})$.

The cases when $\alpha = \alpha_{\min}$ or $\alpha = \alpha_{\max}$ need to be handled separately since we cannot apply Lemma 3. However it can be seen from the ergodic decomposition of such an invariant measure that

$$\sup \left\{ \frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} : \mu \in \mathcal{M}_\sigma(\Sigma), \int f \, \mathrm{d}\mu = \alpha_{\min} \right\}$$

must be the same as

$$\sup \left\{ \frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} : \mu \in \mathcal{E}_\sigma(\Sigma), \int f \, \mathrm{d}\mu = \alpha_{\min} \right\}.$$

If $\mu' \in \mathcal{E}_\sigma(\Sigma)$ occurs in an ergodic decomposition of $\mu \in \mathcal{M}_\sigma(\Sigma)$ where $\int f \, d\mu = \alpha_{min}$, then, by the extremality of α_{min}, $\int f \, d\mu' = \alpha_{min}$. The same is true for $\alpha_{\max}$. This completes the proof of the lower bound.

4. Upper Bound

In this section we shall prove the upper bound in Theorem 1, i.e.

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) \leq \sup_{\mu \in \mathcal{M}_\sigma(\Sigma)} \left\{ \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} : \int f \, d\mu = \alpha, \lambda(\mu, \sigma) > 0 \right\}. \tag{4.1}$$

For $\delta > 0$ let

$$\tilde{\Sigma}(\delta) = \{\omega \in \Sigma : \liminf_{n \to \infty} A_n g(\omega) \geq \delta\}.$$

so that $\tilde{\Sigma}$ can be written as the countable union $\tilde{\Sigma} = \cup_j \tilde{\Sigma}(\delta_j)$, for any sequence $\{\delta_j\}$ with $\lim_{j \to \infty} \delta_j = 0$.

Recall that

$$D_n(\omega) = \operatorname{diam}(T_{\omega_1} \circ \cdots \circ T_{\omega_n}([0,1])) \text{ and } \tilde{\lambda}_n(\omega) = -\frac{1}{n} \log D_n(\omega).$$

An important consequence of Lemma 1 is that for any $\eta > 0$ there exists some $N_0 = N_0(\eta)$ such that

$$\tilde{\lambda}_n(\omega) \geq (1 - \eta) A_n g(\omega) \tag{4.2}$$

for all $n \geq N_0$ and all $\omega \in \tilde{\Sigma}(\delta)$. (To see this, take $\epsilon = \eta\delta$ in the proof of Lemma 1.)

Most of the section will be devoted to proving the following lemma. We consider the sets

$$X(\alpha, N, \rho, \delta) = \left\{ \omega \in \tilde{\Sigma}(\delta) : A_n f(\omega) \in B(\alpha, \rho) \text{ for all } n \geq N \right\},$$

where, for $\rho > 0$, $B(\alpha, \rho) = \{x : |x - \alpha| < \rho\}$.

Lemma 5. *For all $\rho, \epsilon > 0$, $\delta > 0$ sufficiently small and $N \in \mathbb{N}$ we can find a measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ such that $\int f \, d\mu \in B(\alpha, 2\rho)$, $\lambda(\mu, \sigma) > \delta$ and*

$$\dim_H \Pi(X(\alpha, N, \rho, \delta)) \leq \frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} + \epsilon.$$

Before proving this lemma we show how it can be used to deduce (4.1).

Proof of (4.1). First note that, for any fixed $\rho > 0$ and $\delta > 0$, the set $X_\alpha \cap \tilde{\Sigma}(\delta)$ is contained in the increasing union $\cup_{N \in \mathbb{N}} X(\alpha, N, \rho, \delta$ and thus

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}(\delta)) \leq \sup_N \dim_H \Pi(X(\alpha, N, \rho, \delta)).$$

For any $\epsilon, \delta > 0$ and any sequence ρ_n, decreasing to zero as $n \to \infty$, we can use Lemma 5 to find a sequence of invariant measures $\mu_n \in \mathcal{M}_\sigma(\Sigma)$ with $\int f \, d\mu_n \in B(\alpha, 2\rho_n)$, $\lambda(\mu_n, \sigma) > \delta$ and

$$\dim_H \Pi(X_\alpha) \leq \dim_H \Pi(X(\alpha, N_n, \rho_n, \delta)) + \epsilon/2 \leq \frac{h(\mu_n, \sigma)}{\lambda(\mu_n, \sigma)} + \epsilon.$$

If μ_δ is any weak*-limit of μ_n it clearly follows that $\int f \, d\mu_\delta = \alpha$, and from the upper semicontinuity of entropy $\mu \to h(\mu, \sigma)$ (see [15] Theorem 8.2) and the continuity of $\mu \to \lambda(\mu, \sigma)$ it follows that

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}(\delta)) \leq \frac{h(\mu_\delta, \sigma)}{\lambda(\mu_\delta, \sigma)} + \epsilon,$$

where $\lambda(\mu_\delta, \sigma) > \delta$. This holds for any $\delta > 0$ so by taking a countable union of δ_n where $\delta_n \to 0$ we get

$$\dim_H \Pi(X_\alpha \cap \tilde{\Sigma}) \leq \sup_n \left\{ \frac{h(\mu_{\delta_n}, \sigma)}{\lambda(\mu_{\delta_n}, \sigma)} + \epsilon \right\}.$$

Since ϵ was arbitrary, this completes the proof of (4.1). □

Proof of Lemma 5. Fix $\epsilon > 0$, $N \in \mathbb{N}$, $\alpha, \delta, \rho > 0$ and let $X = X(\alpha, N, \rho, \delta)$. By compactness, f and g are uniformly continuous and if N is sufficiently large then for all $n \geq N$,

$$|A_n f(\tau) - A_n f(\omega)| \leq \rho/2 \tag{4.3}$$

whenever $[\omega]_n = [\tau]_n$. Furthermore, by (4.2) we can assume that for all $n \geq N$ and for all $\omega \in X$,

$$A_n g(\omega) \leq (1 + \epsilon) \tilde{\lambda}_n(\omega). \tag{4.4}$$

For $n \geq N$, let Y_n consist of all cylinders $[\omega]_n \in \mathcal{F}_n$ which contain a point in X, i.e. $Y_n = \pi_n(X)$. For each n, define s_n to be the solution to

$$\sum_{[\omega]_n \in Y_n} D_n^{s_n}(\omega) = 1. \tag{4.5}$$

From the definition of Hausdorff dimension, it then immediately follows that

$$\dim_H \Pi(X) \leq \liminf_{n \to \infty} s_n,$$

since the projections of the elements of Y_n form a sequence of covers of $\Pi(X)$ by intervals having diameters decreasing to zero as $n \to \infty$.

Let ν_n be the probability defined by (4.5) on the n-cylinders, i.e.

$$\nu_n([\omega]_n) = D_n^{s_n}(\omega) = e^{-n s_n \tilde{\lambda}_n(\omega)}$$

if $[\omega] \in Y_n$ and zero otherwise. Let μ_n be the corresponding nth level Bernoulli measure defined by $\nu_n = \mu_n|_{\mathcal{F}_n}$. That is, $\mu_n = \nu_n^{\otimes}$ where ν_n is interpreted as a measure in $\mathcal{M}(\mathcal{A}^n)$. It is clear from (4.3) that $\int A_n f \,\mathrm{d}\mu_n \in B(\alpha, 2\rho)$, since each cylinder $[\omega]_n \in Y_n$ only contains $\tau \in \Sigma$ such that $A_n f(\tau) \in B(\alpha, 2\rho)$.

Evaluating the Shannon entropy $H(\nu_n)$ of ν_n gives the equality

$$H(\nu_n) = ns_n \int \tilde{\lambda}_n \,\mathrm{d}\nu_n,$$

where the integral denotes the expected value $\sum_{[\omega]_n} \nu([\omega]_n)\, \tilde{\lambda}_n(\omega)$ of $\tilde{\lambda}_n$ with respect to ν_n. Since $\mu_n = \nu_n^{\otimes}$ and $\tilde{\lambda}_n$ is $\mathcal{F}_n$-measurable, it is easy to see that $H(\nu_n) = h(\mu_n, \sigma)$ and that $\int \tilde{\lambda}_n \,\mathrm{d}\mu_n = \int \tilde{\lambda}_n \,\mathrm{d}\nu_n$.

Thus we have the identity

$$h(\mu_n, \sigma) = s_n \int \tilde{\lambda}_n \,\mathrm{d}\mu_n.$$

In view of (4.4), this means that

$$h(\mu_n, \sigma^n) \geq s_n(1-\epsilon) \int A_n g \,\mathrm{d}\mu_n = (1-\epsilon) s_n \lambda(\mu_n, \sigma^n) \tag{4.6}$$

since μ_n is nth level Bernoulli and σ^n-invariant. Thus

$$s_n \leq \frac{1}{(1-\epsilon)} \frac{h(\mu_n, \sigma^n)}{\lambda(\mu_n, \sigma^n)} \tag{4.7}$$

To complete the proof of Lemma 5, simply note that Lemma 2 implies that $A_n^* \mu_n$ belongs to $\mathcal{E}_\sigma(\Sigma)$ and that $h(A_n^*\mu_n, \sigma) = \frac{1}{n} h(\mu_n, \sigma^n)$ and that $\lambda(A_n^*\mu_n, \sigma) = \frac{1}{n}\lambda(\mu_n, \sigma^n)$. Moreover, $\int A_n f \,\mathrm{d}\nu_n \in B(\alpha, 2\rho)$. □

5. Proof of Theorem 2

We now need to consider the sequences ω where $\liminf_{n\to\infty} A_n g(\omega) = 0$ and we cannot apply Theorem 1. We start by showing that if the limit of the Birkhoff average for such a sequence exists it must necessarily lie in A (see (1.1) for the definition).

Lemma 6. *Let $\{n_j\}_{j\in\mathbb{N}}$ be a subsequence of $\mathbb{N}$. If for any $\omega \in \Sigma$, $\lim_{j\to\infty} A_{n_j} g(\omega) = 0$, then we have that $\lim_{j\to\infty} A_{n_j} f(\omega) \in A$ if the limit exists. In particular, this shows that $\lim_{n\to\infty} A_n g(\omega) = 0$ means $\lim_{n\to\infty} A_n f(\omega) \in A$, if the limit exists.*

Proof. Let $\epsilon, l > 0$. We can find $\delta > 0$ such that $g(\omega) < \delta$ implies that $\mathrm{dist}(f(\omega), A) < \frac{\epsilon}{2}$. If $\frac{1}{n_j} A_{n_j} g(\omega) < \frac{\delta}{l}$ then $g(\sigma^i\omega) < \delta$ for at least $\frac{n_j(l-1)}{l}$ of the values $1 \leq i \leq n_j$.

Thus $\mathrm{dist}(A_{n_j}f(\omega), A) \leq \frac{(l-1)}{l}\epsilon + \frac{K}{l}$, where $K = \sup_\omega \mathrm{dist}(f(\omega), A)$. Since l is arbitrary this completes the proof. □

Hence we know that when $\alpha \notin A$, we do not need to consider those ω which satisfy $\liminf_{n\to\infty} A_n g(\omega) = 0$, since then, if $\lim_{n\to\infty} A_n f(\omega)$ exists, it can only take the values in A. Thus the first part of Theorem 2 follows immediately from Theorem 1.

By renaming the symbols, we may arrange so that

$$A = [a_1, a_2] = [f(\omega_1), f(\omega_2)],$$

where $\omega_1 = (1, 1, 1, \dots)$ and $\omega_2 = (2, 2, 2, \dots)$ correspond to the the indifferent fixed points $x_1 = \Pi(\omega_1)$ and $x_2 = \Pi(\omega_2)$, respectively. We denote by δ_1 and δ_2 the Dirac measures on the sequences $(1, 1, 1, \dots)$ and $(2, 2, 2, \dots)$.

We will need to consider separately the two cases where $\alpha \in (a_1, a_2)$ and when $\alpha \in \{a_1, a_2\}$. For the first case we need the following lemma.

Lemma 7. *For any $\alpha \in (a_1, a_2)$ and $\nu \in \mathcal{E}_\sigma(\Sigma)$ such that $\lambda(\nu, \sigma) > 0$ we can find $\mu \in \mathcal{M}_\sigma(\Sigma)$ such that $\int f \,\mathrm{d}\mu = \alpha$ and*

$$\frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} = \frac{h(\nu, \sigma)}{\lambda(\nu, \sigma)}.$$

Proof. We assume without loss of generality that $\int f \,\mathrm{d}\nu \geq \alpha$. We can then find $p_1, p_2 > 0$ such that $p_1 + p_2 = 1$, $p_2 > 0$ and $p_1 a_1 + p_2 \int f \,\mathrm{d}\nu = \alpha$. We then let $\mu = p_1\delta_1 + p_2\nu$. and deduce that

$$\int f \,\mathrm{d}\mu = p_1 a_1 + p_2 \int f \,\mathrm{d}\mu = \alpha,$$

and

$$\frac{h(\mu, \sigma)}{\lambda(\mu, \sigma)} = \frac{p_2 h(\nu, \sigma)}{p_2 \lambda(\nu, \sigma)} = \frac{h(\nu, \sigma)}{\lambda(\nu, \sigma)}.$$

□

For $\alpha \in (a_1, a_2)$ and $\epsilon > 0$ we take $\nu \in \mathcal{E}_\sigma(\Sigma)$ such that $\frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)} \geq \dim_H \Lambda - \epsilon$. By applying Lemma 7 we can find a measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ such that $\int f \,\mathrm{d}\mu = \alpha$ and $\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)} \geq \dim_H \Lambda - 2\epsilon$. By combining Lemmas 2 and 3 we can find a measure $\eta \in \mathcal{E}_\sigma(\Sigma)$ such that $\int f \,\mathrm{d}\eta = \alpha$ and $\frac{h(\eta,\sigma)}{\lambda(\eta,\sigma)} \geq \dim_H \Lambda - 2\epsilon$. It thus follows that $\eta(\Lambda_\alpha) = 1$ and by using Lemma 4 we can see that

$$\dim_H \Lambda_\alpha \geq \dim_H \Lambda - 2\epsilon.$$

To complete the proof for the second case where $\alpha \in \{a_1, a_2\}$ we follow a similar approach to that used by Gelfert and Rams in [4].

Our strategy is to look at sequences which alternate between a hyperbolic measure of large dimension and the parabolic measure at the fixed point. We arrange that they spend more time at the fixed point and so this will determine the Birkhoff average. However, if the proportion of time at the fixed point does not grow to quickly with relation to the proportion of time described by the hyperbolic measure, then the dimension can be given by the hyperbolic measure.

Without loss of generality, we may assume that $\alpha = a_1$. With the notation introduced before Lemma 7, we study the behaviour about the indifferent fixed point $x_1 = \Pi(\omega_1)$, where $\omega_1 = (1, 1, 1, \dots)$ and where $\int f \, \mathrm{d}\delta_1 = \alpha$ and $\lambda(\delta_1, \sigma) = 0$. We start by taking a measure $\mu \in \mathcal{E}_\sigma(\Sigma)$ such that $\int f \, \mathrm{d}\mu = \beta$ for some $\beta \neq \alpha$ which also satisfies $\lambda(\mu, \sigma) > 0$.

We now combine these two ergodic measures to find a new (non-invariant) measure with high dimension but for which the Birkhoff averages of f tend to α at almost every point. For $\epsilon > 0$ and $N \geq 1$, we define $\Omega(\epsilon, N)$ to be the set of $\omega \in \Sigma$ such that for all $n \geq N$

$$A_n f(\omega) \in B(\beta, \epsilon), \tag{5.1}$$

$$A_n g(\omega) \in B(\lambda(\mu, \sigma), \epsilon), \tag{5.2}$$

$$-\frac{1}{n} \log \mu([\omega_1, \dots, \omega_n]) \in B(h(\mu, \sigma), \epsilon). \tag{5.3}$$

It follows from the Birkhoff Ergodic Theorem, the Shannon–McMillan–Breiman Theorem and Egorov's Theorem that for any fixed $\delta > 0$, we can find a decreasing sequence $\epsilon_i > 0$, such that

$$\mu(\Omega(\epsilon_i', i)) \geq 1 - \delta, \tag{5.4}$$

where $\lim_{i \to \infty} \epsilon_i' = 0$

By the uniformity of the conclusion of Lemma 1 and since $\mathrm{var}_n \, A_n f$ and $\mathrm{var}_n \, A_n g$ uniformly decrease to zero, we can, for $i = 1, 2, \dots$, choose another ϵ_i'' decreasing to zero, so that for all $\omega \in \Sigma$ and all $n \geq i$ we have

$$\mathrm{var}_n \, A_n f(\omega) \leq \epsilon_i'' \tag{5.5}$$

$$\mathrm{var}_n \, A_n g(\omega) \leq \epsilon_i'' \tag{5.6}$$

$$\tilde{\lambda}_n(\omega) \leq A_n g(\omega) + \epsilon_i''. \tag{5.7}$$

Finally, let $\epsilon_i = \max\{\epsilon_i', \epsilon_i''\}$ and let $\Omega(\epsilon_i) = \Omega(\epsilon_i, i)$. Note that ϵ_i, by the construction above, is decreasing. We also need another sequence of integers $\{k_i\}$ such that

$\lim_{i\to\infty} k_i = \infty$ but

$$\lim_{i\to\infty} k_i\epsilon_i = 0. \tag{5.8}$$

For each i we define two measures $\nu_i \in \mathcal{M}(\mathcal{F}^i)$ and $\eta_i \in \mathcal{M}(\mathcal{F}^{ik_i})$ where ν_i simply gives equal weight to any cylinder containing an element of $\Omega(\epsilon_i)$ and η_i is simply the Dirac measure on the cylinder $[1,1,1,\ldots,1]$ of length ik_i. For $q \in \mathbb{N}$ let $n_q = \sum_{i=1}^q i(1+k_i)$. Define the probability $\eta \in \mathcal{M}(\Sigma)$ to be the distribution of a sequence of independent blocks that alternately have distribution ν_i and η_i, for $i = 1,2,3,\ldots$. That is, let

$$\eta = \otimes_{q=1}^{\infty} \left[\nu_q \circ \sigma^{n_{q-1}} \otimes \eta_q \circ \sigma^{n_{q-1}+q}\right].$$

The measure η is not invariant. However, the behaviour of the Birkhoff average $A_n f(\omega)$ of a continuous function f for an η-typical point ω will approach the value $f(\omega_1)$. This is because the proportion of n's, $1 \le n \le N$, such that $\sigma^n\omega$ are close to ω_1 approaches 1 as $N \to \infty$.

Lemma 8.

1. *For η almost all ω we have* $\lim_{n\to\infty} A_n f(\omega) = \alpha$,
2. $\dim_H \eta \circ \Pi^{-1} = \frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}$.

Proof. Note that since f is bounded and

$$\lim_{q\to\infty} \frac{n_q - n_{q-1}}{n_q} = 0 \tag{5.9}$$

to show part 1 we can just consider the limit along the subsequence n_q.

It follows from the definition of η that for η-almost all ω

$$\frac{\sum_{i=1}^q i(\beta - 2\epsilon_i) + \sum_{i=1}^q k_i i(\alpha - 2\epsilon_i)}{n_q} \le A_{n_q} f(\omega)$$
$$\le \frac{\sum_{i=1}^q i(\beta + 2\epsilon_i) + \sum_{i=1}^q k_i i(\alpha + 2\epsilon_i)}{n_q}.$$

The first part then follows since $\lim_{q\to\infty} \frac{\sum_{i=1}^q i}{n_q} = 0$ and $\lim_{q\to\infty} \frac{\sum_{i=i}^q k_i i(\alpha + \epsilon_i)}{n_q} = \alpha$.

For the second part recall that for any probability measure ν on $[0,1]$, if for ν-almost all x

$$\liminf_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log r} \ge s,$$

then $\dim_H \nu \ge s$. (Here $B(x,r) = \{y : |y-x| < r\}$.) We let $\nu = \eta \circ \Pi^{-1}$ and in order to bound the ratio above we will now consider bounds on the quantities $-\log \eta[\omega_1,\ldots,\omega_n]$ and $n\tilde{\lambda}_n(\omega)$.

By using conditions (5.3) and (5.8), we may deduce that the entropy of the distribution of the independent blocks $(\omega_{n_{i-1}}, \omega_{n_{i-1}+1}, \dots, \omega_{n_{i-1}+i-1})$ is at least $\log(1-\delta) + i(h(\mu,\sigma) - \epsilon_i)$. Since we use the uniform distribution, we may deduce that for all ω

$$-\log \eta([\omega_1, \dots, \omega_{n_q}]) \geq q \, \log(1-\delta) + \sum_{i=1}^{q} i(h(\mu,\sigma) - \epsilon_i). \tag{5.10}$$

Note that $q\log(1-\delta)$ is of asymptotic order $o(\sum_{i=1}^q ih(\mu,\sigma))$ for large q.

For estimating the diameters of the cylinders, we note that

$$\begin{aligned}
n_q \tilde{\lambda}_{n_q}(\omega) &\leq \sum_{j=0}^{n_q-1} g(\sigma^j\omega) + n_q\epsilon_q \qquad \text{(by (5.7))} \\
&\leq \sum_{i=1}^{q} \left(\sum_{j=n_{i-1}}^{n_i-1} g(\sigma^j\omega) + \epsilon_i i(1+k_i) \right) \qquad (\text{since } \epsilon_i \geq \epsilon_q) \\
&= \sum_{i=1}^{q} \left(i\, A_i g(\sigma^{n_{i-1}}\omega) + (ik_i)\, A_{ik_i} g(\sigma^{n_{i-1}+i}\omega) + \epsilon_i i(1+k_i) \right). \qquad (5.11)
\end{aligned}$$

Since the $[\sigma^{n_{i-1}}\omega]_i$ contains an ω from $\Omega(\epsilon_i)$ and since $\sigma^{n_{i-1}+i}\omega \in [\omega_0]_{ik_i}$ we obtain, by (5.2) and (5.6), that (5.11) is less than

$$\begin{aligned}
&\sum_{i=1}^{q} \left(i(\lambda(\mu,\sigma) + \epsilon_i + \mathrm{var}_i\, g) + ik_i(A_{ik_i}g(\omega_0) + \mathrm{var}_{ik_i}\, g) + \epsilon_i i(1+k_i) \right) \\
&\leq \sum_{i=1}^{q} i(\lambda(\mu,\sigma) + 3\epsilon_i + 2\epsilon_i k_i). \qquad (5.12)
\end{aligned}$$

Recall that by condition (5.8) $\lim_{i\to\infty} \epsilon_i k_i = 0$.

We now fix $q > 0$ and choose r such that

$$\exp\left(-\sum_{i=1}^{q} i(\lambda(\mu,\sigma) + 3\epsilon_i + 2\epsilon_i k_i) \right) \geq r > \exp\left(-\sum_{i=1}^{q+1} i(\lambda(\mu,\sigma) + 3\epsilon_i + 2\epsilon_i k_i) \right).$$

Consider a ball $B(x,r)$ where $\Pi\omega = x$. It follows from the choice of r above that $B(x,r)$ can intersect at most 3 cylinders of length n_q which carry positive η-measure. Thus, again using the definition of r together with (5.10), we obtain

$$\frac{\log \nu(B(x,r))}{\log r} \geq \frac{\log 3 + q\, \log(1-\delta) + \sum_{i=1}^q i(h(\mu,\sigma) - \epsilon_i)}{\sum_{i=1}^q i(\lambda(\mu,\sigma) + 3\epsilon_i + 2\epsilon_i k_i) + (q+1)(\lambda(\mu,\sigma) + 3\epsilon_{q+1} + 2\epsilon_{q+1}k_{q+1})}.$$

Using condition (5.8) we can observe that this is of the form

$$\frac{-o(q^2) + \sum_{i=1}^q i(h(\mu,\sigma) - o(1))}{o(q^2) + \sum_{i=1}^q i(\lambda(\mu,\sigma) + o(1))}.$$

Since $r \to 0$ as $q \to \infty$, this means for ν almost all x

$$\lim_{r\to 0+} \frac{\log \nu(B(x,r))}{\log r} \geq \frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)},$$

which completes the proof. □

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